BOUNDS IN TERMS OF GÂTEAUX DERIVATIVES FOR THE WEIGHTED $f-$GINI MEAN DIFFERENCE IN LINEAR SPACES

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Abstract. Some bounds in terms of Gâteaux lateral derivatives for the *weighted* $f-$*Gini mean difference* generated by convex and symmetric functions in linear spaces are established. Applications for norms and semi-inner products are also provided.

1. Introduction

For $\mathbf{a}=(a_1,\dots,a_n)\in\mathbb{R}^n$ and $\mathbf{p}=(p_1,\dots,p_n)$ a probability sequence, meaning that $p_i\geq 0$ $(i\in\{1,\dots,n\})$ and $\sum_{i=1}^n p_i=1$, define the $r-$*weighted Gini mean difference,* for $r\in[1,\infty)$, by the formula [1, p. 291]:

$$G_r(\mathbf{p},\mathbf{a}):=\frac{1}{2}\sum_{j=1}^n\sum_{i=1}^n p_ip_j\left|a_i-a_j\right|^r=\sum_{1\leq i<j\leq n}p_ip_j\left|a_i-a_j\right|^r. \tag{1.1}$$

For the uniform probability distribution $\mathbf{p}=\left(\frac{1}{n},\dots,\frac{1}{n}\right)$ we denote

$$G_r(\mathbf{a}):=G_r(\mathbf{p},\mathbf{a})=\frac{1}{2n^2}\sum_{i=1}^n\sum_{j=1}^n\left|a_i-a_j\right|^r=\frac{1}{n^2}\sum_{1\leq i<j\leq n}\left|a_i-a_j\right|^r.$$

For $r=1$ we have the *weighted Gini mean difference* $G(\mathbf{p},\mathbf{a})$, where

$$G(\mathbf{p},\mathbf{a}):=\frac{1}{2}\sum_{j=1}^n\sum_{i=1}^n p_ip_j\left|a_i-a_j\right|=\sum_{1\leq i<j\leq n}p_ip_j\left|a_i-a_j\right|, \tag{1.2}$$

which becomes, for the uniform probability distribution $\mathbf{p}=\left(\frac{1}{n},\dots,\frac{1}{n}\right)$ the *Gini mean difference*

$$G(\mathbf{a}):=\frac{1}{2n^2}\sum_{j=1}^n\sum_{i=1}^n\left|a_i-a_j\right|=\frac{1}{n^2}\sum_{1\leq i<j\leq n}\left|a_i-a_j\right|.$$

For various properties of this and the *Gini index*

$$R(\mathbf{a})=\frac{1}{\bar{a}}G(\mathbf{a}),\ \text{where } \bar{a}:=\frac{1}{n}\sum_{i=1}^n a_i\neq 0,$$

see the papers [6], [7], [1] and [9].

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Now, if we define $\Delta := \{(i,j)\,|i,j \in \{1,\dots,n\}\}$, then we can simply write from (1.1) that

$$G_r(\mathbf{p},\mathbf{a}) = \frac{1}{2}\sum_{(i,j)\in\Delta} p_i p_j |a_i - a_j|^r, \qquad r \geq 1. \tag{1.3}$$

The following result concerning upper and lower bounds for $G_r(\mathbf{p},\mathbf{a})$ may be stated (see [2]):

Theorem 1. *For any $p_i \in (0,1)$, $i \in \{1,\dots,n\}$ with $\sum_{i=1}^n p_i = 1$ and $a_i \in \mathbb{R}$, $i \in \{1,\dots,n\}$, we have the inequalities*

$$\frac{1}{2}\max_{(i,j)\in\Delta}\left\{\frac{p_i^r p_j^r + p_i p_j (1-p_i p_j)^{r-1}}{(1-p_i p_j)^{r-1}} |a_i - a_j|^r\right\} \leq G_r(\mathbf{p},\mathbf{a}) \leq \frac{1}{2}\max_{(i,j)\in\Delta}|a_i - a_j|^r, \tag{1.4}$$

where $r \in [1,\infty)$.

Remark 1. *The case $r = 2$ is of interest, since*

$$G_2(\mathbf{p},\mathbf{a}) = \frac{1}{2}\sum_{(i,j)\in\Delta} p_i p_j |a_i - a_j|^2 = \sum_{i=1}^n p_i a_i^2 - \left(\sum_{i=1}^n p_i a_i\right)^2,$$

for which we can obtain from Theorem 1 the following bounds:

$$\frac{1}{2}\max_{(i,j)\in\Delta}\left\{\frac{p_i p_j}{1-p_i p_j}(a_i - a_j)^2\right\} \leq G_2(\mathbf{p},\mathbf{a}) \leq \frac{1}{2}\max_{(i,j)\in\Delta}(a_i - a_j)^2. \tag{1.5}$$

Remark 2. *Since the function*

$$h_r(t) := \frac{t^r + t(1-t)^{r-1}}{(1-t)^{r-1}} = t + t^r (1-t)^{1-r}$$

defined for $t \in [0,1)$ and $r > 1$ is strictly increasing on $[0,1)$ from Theorem 1 we can obtain a coarser but, perhaps, a more useful lower bound for the $r-$weighted Gini mean difference, namely (see [2]):

$$G_r(\mathbf{p},\mathbf{a}) \geq \frac{1}{2} \cdot \frac{p_m^{2r} + p_m^2 (1-p_m^2)^{r-1}}{(1-p_m^2)^{r-1}} \cdot \max_{(i,j)\in\Delta}|a_i - a_j|^r, \tag{1.6}$$

where p_m is defined above.

For $r = 2$, we then have:

$$G_2(\mathbf{p},\mathbf{a}) \geq \frac{1}{2} \cdot \frac{p_m^2}{1-p_m^2} \cdot \max_{(i,j)\in\Delta}(a_i - a_j)^2. \tag{1.7}$$

For other results related to the above, see the recent paper [2]. For various inequalities concerning $G_2(\mathbf{p},\mathbf{a})$, see the book [4] and the references therein.

The main purpose of the present paper is to provide some bounds in terms of Gâteaux lateral derivatives for the *weighted $f-$Gini mean difference* generated by convex and symmetric functions in linear spaces that has been introduced in the recent work [5] and briefly recalled in the next section. Applications for norms and semi-inner products are also provided.

2. Some Preliminary Results

2.1. **Weighted $f-$Gini Mean Difference.** Consider $f : X \to \mathbb{R}$ be a convex function on the linear space X. Assume that $f(0) = 0$ and f is symmetric, i.e., $f(x) = f(-x)$ for any $x \in X$. In these circumstances we have

$$f(x) = \frac{f(x) + f(-x)}{2} \geq f\left(\frac{x - x}{2}\right) = f(0) = 0$$

showing that f is nonnegative on the entire space X.

For $\mathbf{x} = (x_1, ..., x_n) \in X^n$ and $\mathbf{p} = (p_1, ..., p_n) \in \mathbf{P}^n$ we define the *weighted $f-$Gini mean difference* of the n-tuple $\mathbf{x}$ with the probability distribution $\mathbf{p}$ the positive quantity

$$G_f(\mathbf{p}, \mathbf{x}) := \frac{1}{2} \sum_{i,j=1}^{n} p_i p_j f(x_i - x_j) = \sum_{1 \leq i < j \leq n} p_i p_j f(x_i - x_j) \geq 0. \tag{2.1}$$

For the uniform distribution $\mathbf{u} = \left(\frac{1}{n}, ..., \frac{1}{n}\right) \in \mathbf{P}^n$ we have the *$f-$Gini mean difference* defined by

$$G_f(\mathbf{x}) := \frac{1}{2n^2} \sum_{i,j=1}^{n} f(x_i - x_j) = \frac{1}{n^2} \sum_{1 \leq i < j \leq n} f(x_i - x_j).$$

A natural example of such *$f-$Gini mean difference* can be provided by the convex function $f(x) := \|x\|^r$ with $r \geq 1$ defined on a normed linear space $(X, \|\cdot\|)$. We denote this by

$$G_r(\mathbf{p}, \mathbf{x}) := \frac{1}{2} \sum_{i,j=1}^{n} p_i p_j \|x_i - x_j\|^r = \sum_{1 \leq i < j \leq n} p_i p_j \|x_i - x_j\|^r.$$

Further on, we need to consider another quantity that is naturally related with *$f-$Gini mean difference.* For a convex function $f : X \to \mathbb{R}$ defined on the linear space X with the properties that $f(0) = 0$ define the *mean f-deviation* of an n-tuple of vectors $\mathbf{x} = (x_1, ..., x_n) \in X^n$ with the probability distribution $\mathbf{p} = (p_1, ..., p_n) \in \mathbf{P}^n$ by the non-negative quantity

$$K_f(\mathbf{p}, \mathbf{x}) := \sum_{i=1}^{n} p_i f\left(x_i - \sum_{k=1}^{n} p_k x_k\right). \tag{2.2}$$

The fact that $K_f(\mathbf{p}, \mathbf{x})$ is non-negative follows by Jensen's inequality, namely

$$K_f(\mathbf{p}, \mathbf{x}) \geq f\left(\sum_{i=1}^{n} p_i \left(x_i - \sum_{k=1}^{n} p_k x_k\right)\right) = f(0) = 0.$$

A natural example of such deviations can be provided by the convex function $f(x) := \|x\|^r$ with $r \geq 1$ defined on a normed linear space $(X, \|\cdot\|)$. We denote this by

$$K_r(\mathbf{p}, \mathbf{x}) := \sum_{i=1}^{n} p_i \left\|x_i - \sum_{k=1}^{n} p_k x_k\right\|^r \tag{2.3}$$

and call it the *mean r-absolute deviation* of the n-tuple of vectors $\mathbf{x} = (x_1, ..., x_n) \in X^n$ with the probability distribution $\mathbf{p} = (p_1, ..., p_n) \in \mathbf{P}^n$.

The following connection between the $f-$Gini mean difference and the mean f-deviation holds true:

Theorem 2. *If $f : X \to [0,\infty)$ is a symmetric convex function with $f(0) = 0$, then for any n-tuple of vectors $\mathbf{x} = (x_1, ..., x_n) \in X^n$ and any probability distribution $\mathbf{p} = (p_1, ..., p_n) \in \mathbf{P}^n$ we have the inequalities*

$$G_f(\mathbf{p},\mathbf{x}) \geq \frac{1}{2} K_f(\mathbf{p},\mathbf{x}) \geq G_f\left(\mathbf{p}, \frac{1}{2}\mathbf{x}\right). \tag{2.4}$$

Both inequalities in (2.4) are sharp and the constant $\frac{1}{2}$ best possible.

The following particular case for norms is of interest due to its natural generalization for the scalar case that is used in applications:

Corollary 1. *Let $(X, \|\cdot\|)$ be a normed space. Then for any n-tuple of vectors $\mathbf{x} = (x_1, ..., x_n) \in X^n$ and any probability distribution $\mathbf{p} = (p_1, ..., p_n) \in \mathbf{P}^n$ we have*

$$G_r(\mathbf{p},\mathbf{x}) \geq \frac{1}{2} K_r(\mathbf{p},\mathbf{x}) \geq \frac{1}{2^r} G_r(\mathbf{p},\mathbf{x}) \tag{2.5}$$

or, equivalently,

$$\sum_{i,j=1}^{n} p_i p_j \|x_i - x_j\|^r \geq \sum_{i=1}^{n} p_i \left\| x_i - \sum_{k=1}^{n} p_k x_k \right\|^r \geq \frac{1}{2^{r-1}} \sum_{i,j=1}^{n} p_i p_j \|x_i - x_j\|^r \tag{2.6}$$

for any $r \geq 1$.

Remark 3. *By symmetrie reasons we have*

$$\sum_{i,j=1}^{n} p_i p_j \|x_i - x_j\|^r = 2 \sum_{1 \leq i < j \leq n} p_i p_j \|x_i - x_j\|^r$$

and since

$$\sum_{1 \leq i < j \leq n} p_i p_j = \frac{1}{2}\left(\sum_{i,j=1}^{n} p_i p_j - \sum_{i=1}^{n} p_i^2\right) = \frac{1}{2}\left(1 - \sum_{i=1}^{n} p_i^2\right) = \frac{1}{2}\sum_{i=1}^{n} p_i (1 - p_i)$$

then we may state from (2.6) the following simpler inequality:

$$\begin{aligned} \sum_{i=1}^{n} p_i (1 - p_i) \max_{1 \leq i < j \leq n} \|x_i - x_j\|^r &\geq \sum_{i=1}^{n} p_i \left\| x_i - \sum_{k=1}^{n} p_k x_k \right\|^r \\ &\geq \frac{1}{2^{r-1}} \sum_{i=1}^{n} p_i (1 - p_i) \min_{1 \leq i < j \leq n} \|x_i - x_j\|^r. \end{aligned} \tag{2.7}$$

2.2. **The Gâteaux Derivatives of Convex Functions.** Assume that $f : X \to \mathbb{R}$ is a *convex function* on the real linear space X. Since for any vectors $x, y \in X$ the function $g_{x,y} : \mathbb{R} \to \mathbb{R}$, $g_{x,y}(t) := f(x + ty)$ is convex it follows that the following limits exist

$$\nabla_{+(-)} f(x)(y) := \lim_{t \to 0+(-)} \frac{f(x + ty) - f(x)}{t}$$

and they are called the *right(left) Gâteaux derivatives* of the function f in the point x over the direction y.

It is obvious that for any $t > 0 > s$ we have

$$\frac{f(x+ty)-f(x)}{t} \geq \nabla_{+}f(x)(y) = \inf_{t>0}\left[\frac{f(x+ty)-f(x)}{t}\right] \geq \sup_{s<0}\left[\frac{f(x+sy)-f(x)}{s}\right] = \nabla_{-}f(x)(y) \geq \frac{f(x+sy)-f(x)}{s} \tag{2.8}$$

for any $x, y \in X$ and, in particular,

$$\nabla_{-}f(u)(u-v) \geq f(u) - f(v) \geq \nabla_{+}f(v)(u-v) \tag{2.9}$$

for any $u, v \in X$. We call this *the gradient inequality* for the convex function f. It will be used frequently in the sequel in order to obtain various results related to Jensen's inequality.

The following properties are also of importance:

$$\nabla_{+}f(x)(-y) = -\nabla_{-}f(x)(y), \tag{2.10}$$

and

$$\nabla_{+(-)}f(x)(\alpha y) = \alpha\nabla_{+(-)}f(x)(y) \tag{2.11}$$

for any $x, y \in X$ and $\alpha \geq 0$.

The right Gâteaux derivative is *subadditive* while the left one is *superadditive*, i.e.,

$$\nabla_{+}f(x)(y+z) \leq \nabla_{+}f(x)(y) + \nabla_{+}f(x)(z) \tag{2.12}$$

and

$$\nabla_{-}f(x)(y+z) \geq \nabla_{-}f(x)(y) + \nabla_{-}f(x)(z) \tag{2.13}$$

for any $x, y, z \in X$.

Some natural examples can be provided by the use of normed spaces.

Assume that $(X, \|\cdot\|)$ is a real normed linear space. The function $f : X \to \mathbb{R}$, $f(x) := \frac{1}{2}\|x\|^2$ is a convex function which generates *the superior* and *the inferior semi-inner products*

$$\langle y, x\rangle_{s(i)} := \lim_{t\to 0+(-)} \frac{\|x+ty\|^2 - \|x\|^2}{t}.$$

For a comprehensive study of the properties of these mappings in the Geometry of Banach Spaces see the monograph [3].

For the convex function $f_p : X \to \mathbb{R}$, $f_p(x) := \|x\|^p$ with $p > 1$, we have

$$\nabla_{+(-)}f_p(x)(y) = \begin{cases} p\|x\|^{p-2}\langle y, x\rangle_{s(i)} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

for any $y \in X$.

If $p = 1$, then we have

$$\nabla_{+(-)}f_1(x)(y) = \begin{cases} \|x\|^{-1}\langle y, x\rangle_{s(i)} & \text{if } x \neq 0, \\ +(-)\|y\| & \text{if } x = 0, \end{cases}$$

for any $y \in X$.

This class of functions will be used to illustrate the inequalities obtained in the general case of convex functions defined on an entire linear space.

The following result for the general case of convex functions holds (see [5]):

Theorem 3. *Let $f : X \to \mathbb{R}$ be a convex function. Then for any $x, y \in X$ and $t \in [0,1]$ we have*

$$\begin{aligned} t(1-t)\left[\nabla_{-}f(y)(y-x) - \nabla_{+}f(x)(y-x)\right] \\ \geq tf(x) + (1-t)f(y) - f(tx+(1-t)y) \\ \geq t(1-t)\left[\nabla_{+}f(tx+(1-t)y)(y-x) - \nabla_{-}f(tx+(1-t)y)(y-x)\right] \geq 0. \end{aligned} \tag{2.14}$$

The following particular case for norms may be stated:

Corollary 2. *If x and y are two vectors in the normed linear space $(X, \|\cdot\|)$ such that $0 \notin [x,y] := \{(1-s)x + sy, s \in [0,1]\}$, then for any $p \geq 1$ we have the inequalities*

$$\begin{aligned} pt(1-t)\left[\|y\|^{p-2}\langle y-x, y\rangle_i - \|x\|^{p-2}\langle y-x, x\rangle_s\right] \\ \geq t\|x\|^p + (1-t)\|y\|^p - \|tx+(1-t)y\|^p \\ \geq pt(1-t)\|tx+(1-t)y\|^{p-2}\left[\langle y-x, tx+(1-t)y\rangle_s - \langle y-x, tx+(1-t)y\rangle_i\right] \geq 0 \end{aligned} \tag{2.15}$$

for any $t \in [0,1]$. If $p \geq 2$ the inequality holds for any x and y.

Remark 4. *If the normed space $(X, \|\cdot\|)$ is smooth and the norm generated by the semi-inner product $[\cdot,\cdot] : X \times X \to \mathbb{R}$, then the inequality (2.15) can be written as*

$$\begin{aligned} pt(1-t)\left\{\left[y-x, \|y\|^{p-2}y\right] - \left[y-x, \|x\|^{p-2}x\right]\right\} \\ \geq t\|x\|^p + (1-t)\|y\|^p - \|tx+(1-t)y\|^p \end{aligned} \tag{2.16}$$

for any $t \in [0,1]$.

Moreover, if $(X, \langle\cdot,\cdot\rangle)$ is an inner product space, then (2.16) becomes

$$\begin{aligned} pt(1-t)\left\langle y-x, \|y\|^{p-2}y - \|x\|^{p-2}x\right\rangle \\ \geq t\|x\|^p + (1-t)\|y\|^p - \|tx+(1-t)y\|^p \end{aligned} \tag{2.17}$$

for any $t \in [0,1]$.

3. Bounds in Terms of Gâteaux Derivatives

The following result in which we provide some upper and lower bounds for the nonnegative quantity

$$G_f(\mathbf{p}, \mathbf{x}) - \frac{1}{2}K_f(\mathbf{p}, \mathbf{x})$$

considered in Theorem 2 may be stated:

Theorem 4. *If $f : X \to \mathbb{R}$ is a symmetric convex function with $f(0) = 0$, then for any n-tuple of vectors $\mathbf{x} = (x_1, ..., x_n) \in X^n$ and any probability distribution*

$\mathbf{p} = (p_1, ..., p_n) \in \mathbf{P}^n$ *we have the inequalities*

$$\begin{aligned}
&\frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n} p_i p_j \nabla_- f(x_i - x_j)\left(\sum_{k=1}^{n} p_k x_k - x_j\right) \\
&\geq G_f(\mathbf{p}, \mathbf{x}) - \frac{1}{2} K_f(\mathbf{p}, \mathbf{x}) \\
&\geq \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n} p_i p_j \nabla_+ f\left(\sum_{k=1}^{n} p_k x_k - x_i\right)\left(x_j - \sum_{k=1}^{n} p_k x_k\right) \geq 0.
\end{aligned} \tag{3.1}$$

Proof. Utilising the gradient inequality (2.9) we have

$$\begin{aligned}
&\nabla_- f(x_i - x_j)\left(\sum_{k=1}^{n} p_k x_k - x_j\right) \\
&\geq f(x_i - x_j) - f\left(x_i - \sum_{k=1}^{n} p_k x_k\right) \\
&\geq \nabla_+ f\left(x_i - \sum_{k=1}^{n} p_k x_k\right)\left(\sum_{k=1}^{n} p_k x_k - x_j\right)
\end{aligned} \tag{3.2}$$

for any $i, j \in \{1, ..., n\}$.

By the symmetrie of the function f and the subadditivity of the Gâteaux derivative $\nabla_+ f(\cdot)(\cdot)$ in the second variable we also have

$$\begin{aligned}
&\nabla_+ f\left(x_i - \sum_{k=1}^{n} p_k x_k\right)\left(\sum_{k=1}^{n} p_k x_k - x_j\right) \\
&= \nabla_+ f\left(\sum_{k=1}^{n} p_k x_k - x_i\right)\left(x_j - \sum_{k=1}^{n} p_k x_k\right) \\
&\geq \nabla_+ f\left(\sum_{k=1}^{n} p_k x_k - x_i\right)(x_j) - \nabla_+ f\left(\sum_{k=1}^{n} p_k x_k - x_i\right)\left(\sum_{k=1}^{n} p_k x_k\right)
\end{aligned} \tag{3.3}$$

for any $i, j \in \{1, ..., n\}$.

Utilising (3.2) and (3.3) we may state that

$$\begin{aligned}
&\nabla_- f(x_i - x_j)\left(\sum_{k=1}^{n} p_k x_k - x_j\right) \\
&\geq f(x_i - x_j) - f\left(x_i - \sum_{k=1}^{n} p_k x_k\right) \\
&\geq \nabla_+ f\left(\sum_{k=1}^{n} p_k x_k - x_i\right)\left(x_j - \sum_{k=1}^{n} p_k x_k\right) \\
&\geq \nabla_+ f\left(\sum_{k=1}^{n} p_k x_k - x_i\right)(x_j) - \nabla_+ f\left(\sum_{k=1}^{n} p_k x_k - x_i\right)\left(\sum_{k=1}^{n} p_k x_k\right)
\end{aligned} \tag{3.4}$$

for any $i, j \in \{1, ..., n\}$.

Now, if we multiply the inequality with $p_j \geq 0$ and sum over j from 1 to n we get

$$
\begin{aligned}
(3.5)\qquad & \sum_{j=1}^{n} p_j \nabla_- f(x_i - x_j)\left(\sum_{k=1}^{n} p_k x_k - x_j\right) \\
\geq\ & \sum_{j=1}^{n} p_j f(x_i - x_j) - f\left(x_i - \sum_{k=1}^{n} p_k x_k\right) \\
\geq\ & \sum_{j=1}^{n} p_j \nabla_+ f\left(\sum_{k=1}^{n} p_k x_k - x_i\right)\left(x_j - \sum_{k=1}^{n} p_k x_k\right) \\
\geq\ & \sum_{j=1}^{n} p_j \nabla_+ f\left(\sum_{k=1}^{n} p_k x_k - x_i\right)(x_j) - \nabla_+ f\left(\sum_{k=1}^{n} p_k x_k - x_i\right)\left(\sum_{k=1}^{n} p_k x_k\right) \\
\geq\ & 0
\end{aligned}
$$

where the last inequality follows by the subadditivity of the function

$$\nabla_+ f\left(\sum_{k=1}^{n} p_k x_k - x_i\right)(\cdot) \text{ with } i \in \{1, ..., n\}.$$

Finally, if we multiply the inequality (3.5) with $p_i \geq 0$ and sum over i from 1 to n we get the desired result (3.1). $\square$

The following particular case for norm holds:

Corollary 3. *Let $(X, \|\cdot\|)$ be a normed space. Then for an n-tuple of vectors $\mathbf{x} = (x_1, ..., x_n) \in X^n$ and the probability distribution $\mathbf{p} = (p_1, ..., p_n) \in \mathbf{P}^n$ we have the inequalities:*

$$
\begin{aligned}
(3.6)\qquad & r \sum_{j=1}^{n} \sum_{l=1}^{n} p_l p_j \|x_l - x_j\|^{r-2} \left\langle \sum_{k=1}^{n} p_k x_k - x_j, x_l - x_j \right\rangle_i \\
& \geq \sum_{l,j=1}^{n} p_l p_j \|x_l - x_j\|^r - \sum_{l=1}^{n} p_l \left\| x_l - \sum_{k=1}^{n} p_k x_k \right\|^r \\
& \geq r \sum_{j=1}^{n} \sum_{l=1}^{n} p_l p_j \left\| \sum_{k=1}^{n} p_k x_k - x_l \right\|^{r-2} \left\langle x_j - \sum_{k=1}^{n} p_k x_k, \sum_{k=1}^{n} p_k x_k - x_l \right\rangle_s \geq 0.
\end{aligned}
$$

If $r \geq 2$ then we have no restiction for $\mathbf{x}$ and $\mathbf{p}$. If $r \in [1, 2)$ then we need to assume that $x_l - x_j \neq 0$ and $\sum_{k=1}^{n} p_k x_k - x_l \neq 0$ for all $l, j \in \{1, ..., n\}$.

Remark 5. *The case $r = 2$ produces the following simpler inequality*

$$
\begin{aligned}
(3.7)\qquad & 2 \sum_{j=1}^{n} \sum_{l=1}^{n} p_l p_j \left\langle \sum_{k=1}^{n} p_k x_k - x_j, x_l - x_j \right\rangle_i \\
& \geq \sum_{l,j=1}^{n} p_l p_j \|x_l - x_j\|^2 - \sum_{l=1}^{n} p_l \left\| x_l - \sum_{k=1}^{n} p_k x_k \right\|^2 \\
& \geq 2 \sum_{j=1}^{n} \sum_{l=1}^{n} p_l p_j \left\langle x_j - \sum_{k=1}^{n} p_k x_k, \sum_{k=1}^{n} p_k x_k - x_l \right\rangle_s \geq 0.
\end{aligned}
$$

that holds for any n-tuple of vectors $\mathbf{x} = (x_1, ..., x_n) \in X^n$ and any probability distribution $\mathbf{p} = (p_1, ..., p_n) \in \mathbf{P}^n$.

Remark 6. *If the normed space $(X, \|\cdot\|)$ is smooth and the norm generated by the semi-inner product $[\cdot, \cdot] : X \times X \to \mathbb{R}$, then the inequality (3.7) can be written as*

$$2 \sum_{j=1}^{n} \sum_{l=1}^{n} p_l p_j \left[\sum_{k=1}^{n} p_k x_k - x_j, x_l - x_j \right] \geq \sum_{l,j=1}^{n} p_l p_j \|x_l - x_j\|^2 - \sum_{l=1}^{n} p_l \left\| x_l - \sum_{k=1}^{n} p_k x_k \right\|^2 \geq 0. \tag{3.8}$$

Further on we provide upper and lower bounds for the nonnegative quantity considered in the second part of Theorem 2, namely:

$$\frac{1}{2} K_f (\mathbf{p}, \mathbf{x}) - G_f \left(\mathbf{p}, \frac{1}{2} \mathbf{x} \right).$$

Theorem 5. *If $f : X \to \mathbb{R}$ is a symmetric convex function with $f(0) = 0$, then for any n-tuple of vectors $\mathbf{x} = (x_1, ..., x_n) \in X^n$ and any probability distribution $\mathbf{p} = (p_1, ..., p_n) \in \mathbf{P}^n$ we have the inequalities*

$$\begin{aligned} &\frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{n} p_i p_j \nabla_{-} f \left(\sum_{k=1}^{n} p_k x_k - x_j \right) \left(\sum_{k=1}^{n} p_k x_k - \frac{x_i + x_j}{2} \right) \\ &\geq \frac{1}{2} K_f (\mathbf{p}, \mathbf{x}) - G_f \left(\mathbf{p}, \frac{1}{2} \mathbf{x} \right) \\ &\geq \frac{1}{4} \left[\sum_{j=1}^{n} \sum_{i=1}^{n} p_i p_j \nabla_{+} f \left(\frac{x_i - x_j}{2} \right) \left(\sum_{k=1}^{n} p_k x_k - \frac{x_i + x_j}{2} \right) \right. \\ &\left. - \sum_{j=1}^{n} \sum_{i=1}^{n} p_i p_j \nabla_{-} f \left(\frac{x_i - x_j}{2} \right) \left(\sum_{k=1}^{n} p_k x_k - \frac{x_i + x_j}{2} \right) \right] \geq 0. \end{aligned} \tag{3.9}$$

Proof. Consider the inequality (2.14) for $t = \frac{1}{2}$ to get

$$\begin{aligned} \frac{1}{4} \left[\nabla_{-} f(y)(y - x) - \nabla_{+} f(x)(y - x) \right] &\geq \frac{f(x) + f(y)}{2} - f\left(\frac{x + y}{2} \right) \\ &\geq \frac{1}{4} \left[\nabla_{+} f\left(\frac{x + y}{2} \right)(y - x) - \nabla_{-} f\left(\frac{x + y}{2} \right)(y - x) \right] \geq 0 \end{aligned} \tag{3.10}$$

for any $x, y \in X$.

Now, if in (3.10) we choose $x = x_i - \sum_{k=1}^{n} p_k x_k$ and $y = \sum_{k=1}^{n} p_k x_k - x_j$ with $i, j \in \{1, ..., n\}$ and take into account the symmetrie of the function f, then we have

$$\begin{aligned}
(3.11)\qquad & \frac{1}{2}\left[\nabla_{-} f\left(\sum_{k=1}^{n} p_k x_k - x_j\right)\left(\sum_{k=1}^{n} p_k x_k - \frac{x_i + x_j}{2}\right)\right. \\
& \left. -\nabla_{+} f\left(x_i - \sum_{k=1}^{n} p_k x_k\right)\left(\sum_{k=1}^{n} p_k x_k - \frac{x_i + x_j}{2}\right)\right] \\
& \geq \frac{1}{2}\left[f\left(x_i - \sum_{k=1}^{n} p_k x_k\right) + f\left(x_j - \sum_{k=1}^{n} p_k x_k\right)\right] - f\left(\frac{1}{2}(x_i - x_j)\right) \\
& \geq \frac{1}{2}\left[\nabla_{+} f\left(\frac{x_i - x_j}{2}\right)\left(\sum_{k=1}^{n} p_k x_k - \frac{x_i + x_j}{2}\right)\right. \\
& \left. -\nabla_{-} f\left(\frac{x_i - x_j}{2}\right)\left(\sum_{k=1}^{n} p_k x_k - \frac{x_i + x_j}{2}\right)\right] \geq 0
\end{aligned}$$

for any $i, j \in \{1, ..., n\}$.

Further on, if we multiply (3.11) with $p_i p_j \geq 0$ and sum over i and j from 1 to n we deduce

$$\begin{aligned}
(3.12)\qquad & \frac{1}{4}\left[\sum_{j=1}^{n}\sum_{i=1}^{n} p_i p_j \nabla_{-} f\left(\sum_{k=1}^{n} p_k x_k - x_j\right)\left(\sum_{k=1}^{n} p_k x_k - \frac{x_i + x_j}{2}\right)\right. \\
& \left. -\sum_{j=1}^{n}\sum_{i=1}^{n} p_i p_j \nabla_{+} f\left(x_i - \sum_{k=1}^{n} p_k x_k\right)\left(\sum_{k=1}^{n} p_k x_k - \frac{x_i + x_j}{2}\right)\right] \\
& \geq \frac{1}{2} K_f(\mathbf{p}, \mathbf{x}) - G_f\left(\mathbf{p}, \frac{1}{2}\mathbf{x}\right) \\
& \geq \frac{1}{4}\left[\sum_{j=1}^{n}\sum_{i=1}^{n} p_i p_j \nabla_{+} f\left(\frac{x_i - x_j}{2}\right)\left(\sum_{k=1}^{n} p_k x_k - \frac{x_i + x_j}{2}\right)\right. \\
& \left. -\sum_{j=1}^{n}\sum_{i=1}^{n} p_i p_j \nabla_{-} f\left(\frac{x_i - x_j}{2}\right)\left(\sum_{k=1}^{n} p_k x_k - \frac{x_i + x_j}{2}\right)\right] \geq 0.
\end{aligned}$$

By the symmetrie of the function and the symmetrie of summation we have

$$\begin{aligned}
(3.13)\qquad & \sum_{j=1}^{n}\sum_{i=1}^{n} p_i p_j \nabla_{+} f\left(x_i - \sum_{k=1}^{n} p_k x_k\right)\left(\sum_{k=1}^{n} p_k x_k - \frac{x_i + x_j}{2}\right) \\
= & \sum_{j=1}^{n}\sum_{i=1}^{n} p_i p_j \nabla_{+} f\left(x_j - \sum_{k=1}^{n} p_k x_k\right)\left(\sum_{k=1}^{n} p_k x_k - \frac{x_i + x_j}{2}\right) \\
= & \sum_{j=1}^{n}\sum_{i=1}^{n} p_i p_j \nabla_{+} f\left(\sum_{k=1}^{n} p_k x_k - x_j\right)\left(\frac{x_i + x_j}{2} - \sum_{k=1}^{n} p_k x_k\right) \\
= & -\sum_{j=1}^{n}\sum_{i=1}^{n} p_i p_j \nabla_{-} f\left(\sum_{k=1}^{n} p_k x_k - x_j\right)\left(\sum_{k=1}^{n} p_k x_k - \frac{x_i + x_j}{2}\right).
\end{aligned}$$

Finally, on utilising the relations (3.12) and (3.13) we deuce the desired result (3.9). □

The following particular case for norms can be stated:

Corollary 4. *Let $(X, \|\cdot\|)$ be a normed space. Then for an n-tuple of vectors $\mathbf{x} = (x_1, ..., x_n) \in X^n$ and the probability distribution $\mathbf{p} = (p_1, ..., p_n) \in \mathbf{P}^n$ we have the inequalities:*

$$
\begin{aligned}
(3.14)\qquad & r\sum_{j=1}^{n}\sum_{l=1}^{n} p_l p_j \left\|\sum_{k=1}^{n} p_k x_k - x_j\right\|^{r-2} \left\langle \sum_{k=1}^{n} p_k x_k - \frac{x_l + x_j}{2}, \sum_{k=1}^{n} p_k x_k - x_j \right\rangle_i \\
& \geq \sum_{i=1}^{n} p_i \left\| x_i - \sum_{k=1}^{n} p_k x_k \right\|^r - \frac{1}{2^{r-1}} \sum_{i,j=1}^{n} p_i p_j \|x_i - x_j\|^r \\
& \geq \frac{1}{2^r} r \sum_{j=1}^{n}\sum_{l=1}^{n} p_l p_j \|x_l - x_j\|^{r-2} \left[\left\langle \sum_{k=1}^{n} p_k x_k - \frac{x_l + x_j}{2}, x_l - x_j \right\rangle_s \right. \\
& \left. - \left\langle \sum_{k=1}^{n} p_k x_k - \frac{x_l + x_j}{2}, x_l - x_j \right\rangle_i \right] \geq 0.
\end{aligned}
$$

If $r \geq 2$ then we have no restiction for $\mathbf{x}$ and $\mathbf{p}$. If $r \in [1, 2)$ then we need to assume that $x_l - x_j \neq 0$ and $\sum_{k=1}^{n} p_k x_k - x_j \neq 0$ for all $l, j \in \{1, ..., n\}$.

Remark 7. *The case $r = 2$ is of interest since produces a much simpler inequality*

$$
\begin{aligned}
(3.15)\qquad & 2\sum_{j=1}^{n}\sum_{l=1}^{n} p_l p_j \left\langle \sum_{k=1}^{n} p_k x_k - \frac{x_l + x_j}{2}, \sum_{k=1}^{n} p_k x_k - x_j \right\rangle_i \\
& \geq \sum_{i=1}^{n} p_i \left\| x_i - \sum_{k=1}^{n} p_k x_k \right\|^2 - \frac{1}{2} \sum_{i,j=1}^{n} p_i p_j \|x_i - x_j\|^2 \\
& \geq \frac{1}{2} \sum_{j=1}^{n}\sum_{l=1}^{n} p_l p_j \left[\left\langle \sum_{k=1}^{n} p_k x_k - \frac{x_l + x_j}{2}, x_l - x_j \right\rangle_s \right. \\
& \left. - \left\langle \sum_{k=1}^{n} p_k x_k - \frac{x_l + x_j}{2}, x_l - x_j \right\rangle_i \right] \geq 0
\end{aligned}
$$

that holds for any n-tuple of vectors $\mathbf{x} = (x_1, ..., x_n) \in X^n$ and any probability distribution $\mathbf{p} = (p_1, ..., p_n) \in \mathbf{P}^n$.

Remark 8. *f the normed space $(X, \|\cdot\|)$ is smooth and the norm generated by the semi-inner product $[\cdot, \cdot] : X \times X \to \mathbb{R}$, then the inequality (3.15) can be written as*

$$
\begin{aligned}
(3.16)\qquad & 2\sum_{j=1}^{n}\sum_{l=1}^{n} p_l p_j \left[\sum_{k=1}^{n} p_k x_k - \frac{x_l + x_j}{2}, \sum_{k=1}^{n} p_k x_k - x_j \right] \\
& \geq \sum_{i=1}^{n} p_i \left\| x_i - \sum_{k=1}^{n} p_k x_k \right\|^2 - \frac{1}{2} \sum_{i,j=1}^{n} p_i p_j \|x_i - x_j\|^2 \geq 0.
\end{aligned}
$$

4. Other Bounds

In [5] we also established the following upper bound for the weighted $f-$Gini mean difference:

Theorem 6. *Assume that $f : X \to \mathbb{R}$ is a symmetric convex function with $f(0) = 0$. If x and y are two vectors and $t \in [0,1]$ with $(1-t)x + ty = 0$ then for any n-tuple of vectors $\mathbf{x} = (x_1, ..., x_n) \in X^n$ with the property that $x_i - x_j \in [x,y]$ for all $i,j \in \{1,...,n\}$ we have the inequality*

$$\frac{1}{2}\left[(1-t)f(x) + tf(y)\right] \geq G_f(\mathbf{p},\mathbf{x}), \tag{4.1}$$

for any probability distribution $\mathbf{p} = (p_1, ..., p_n) \in \mathbf{P}^n$.

It is thus natural to ask for an uper bound for the positive quantity

$$\frac{1}{2}\left[(1-t)f(x) + tf(y)\right] - G_f(\mathbf{p},\mathbf{x}).$$

The following result holds:

Theorem 7. *Assume that $f : X \to \mathbb{R}$ is a symmetric convex function with $f(0) = 0$. If x and y are two vectors and $t \in [0,1]$ with $(1-t)x + ty = 0$ then for any n-tuple of vectors $\mathbf{x} = (x_1, ..., x_n) \in X^n$ with the property that $x_i - x_j \in [x,y]$ for all $i,j \in \{1,...,n\}$ we have the inequality*

$$\begin{aligned} 0 &\leq \frac{1}{2}\left[(1-t)f(x) + tf(y)\right] - G_f(\mathbf{p},\mathbf{x}) \\ &\leq \frac{1}{8}\left[\nabla_{-}f(y)(y-x) - \nabla_{+}f(x)(y-x)\right], \end{aligned} \tag{4.2}$$

for any probability distribution $\mathbf{p} = (p_1, ..., p_n) \in \mathbf{P}^n$.

Proof. Since $x_i - x_j \in [x,y]$ for $i,j \in \{1,...,n\}$, then there exists the numbers $t_{ij} \in [0,1]$ such that $x_i - x_j = (1-t_{ij})x + t_{ij}y$ for $i,j \in \{1,...,n\}$.

Let $\mathbf{p} = (p_1, ..., p_n) \in \mathbf{P}^n$. Then by the above equality we get that

$$p_i p_j (x_i - x_j) = (1-t_{ij}) p_i p_j x + t_{ij} p_i p_j y$$

for any $i,j \in \{1,...,n\}$. If we sum over i,j from 1 to n, then we get

$$\begin{aligned} 0 = \sum_{i,j=1}^{n} p_i p_j (x_i - x_j) &= \sum_{i,j=1}^{n} \left[(1-t_{ij}) p_i p_j x + t_{ij} p_i p_j y\right] \\ &= \left(1 - \sum_{i,j=1}^{n} t_{ij} p_i p_j\right) x + \left(\sum_{i,j=1}^{n} t_{ij} p_i p_j\right) y. \end{aligned} \tag{4.3}$$

Now, due to the fact that $(1-t)x + ty = 0$ and the representation is unique, we get that $t = \sum_{i,j=1}^{n} t_{ij} p_i p_j$.

On the other hand we have (see

$$\begin{aligned} t_{ij}(1-t_{ij})&\left[\nabla_{-}f(y)(y-x) - \nabla_{+}f(x)(y-x)\right] \\ &\geq t_{ij} f(x) + (1-t_{ij}) f(y) - f\left[t_{ij} x + (1-t_{ij}) y\right] \\ &= t_{ij} f(x) + (1-t_{ij}) f(y) - f(x_i - x_j). \end{aligned} \tag{4.4}$$

Now, if we multiply (4.4) by $p_i p_j \geq 0$, sum over i and j from 1 to n and divide by 2, then we get

$$\frac{1}{2}\left[\nabla_{-} f(y)(y-x)-\nabla_{+} f(x)(y-x)\right] \sum_{i,j=1}^{n} p_i p_j t_{ij}\left(1-t_{ij}\right) \geq \frac{1}{2}\left[(1-t) f(x)+t f(y)\right]-G_f(\mathbf{p}, \mathbf{x}), \tag{4.5}$$

which is an interesting inequality in itself provided that one knows the parameters t_{ij} for any $i, j \in \{1, ..., n\}$.

In the case that these are not known, since $t_{ij}(1-t_{ij}) \leq \frac{1}{4}$ for any $i, j \in \{1, ..., n\}$, then

$$\sum_{i,j=1}^{n} p_i p_j t_{ij}\left(1-t_{ij}\right) \leq \frac{1}{4},$$

which together with (4.5) provides the desired result (4.2). □

The following particular case for norms is of interest:

Corollary 5. *Let $(X, \|\cdot\|)$ be a normed space. If x and y are two nonzero vectors and $t \in [0,1]$ with $(1-t)x + ty = 0$ then for any n-tuple of vectors $\mathbf{x} = (x_1, ..., x_n) \in X^n$ with the property that $x_i - x_j \in [x, y]$ for all $i, j \in \{1, ..., n\}$ we have the inequality*

$$0 \leq \frac{1}{2}\left[(1-t)\|x\|^r + t\|y\|^r\right] - G_r(\mathbf{p}, \mathbf{x}) \leq \frac{1}{8} r\left[\langle y-x, y\rangle_i \|y\|^{r-2} - \langle y-x, x\rangle_s \|x\|^{r-2}\right], \tag{4.6}$$

for any probability distribution $\mathbf{p} = (p_1, ..., p_n) \in \mathbf{P}^n$ and $r \geq 1$.

Remark 9. *We observe that if $(X, \langle\cdot,\cdot\rangle)$ is an inner product space, then the inequality (4.7) has a simpler form, namely*

$$0 \leq \frac{1}{2}\left[(1-t)\|x\|^r + t\|y\|^r\right] - G_r(\mathbf{p}, \mathbf{x}) \leq \frac{1}{8} r\left\langle y-x, \|y\|^{r-2} y - \|x\|^{r-2} x\right\rangle, \tag{4.7}$$

for any probability distribution $\mathbf{p} = (p_1, ..., p_n) \in \mathbf{P}^n$ and $r \geq 1$.

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